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# Load Updating for Finite Element Models

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**A method for reducing the differences between experimental testing or real-world events and their finite element counterparts is presented. In this method system identification or finite element model updating are not performed; instead, the loads on the finite element model that would create equivalent displacements or strains are identified by various means, thus saving a large amount of computational effort. This method of finding an applied load is also extremely useful in the analysis of testing data for vehicles and other structures. Using methods based on classical least-squares methods, we present the basis for finite element load updating and sample application through the use of a computer program for a beam under static loads. We examine the conditioning number of the resulting system of equations and provide a least-squares solution that is more robust through the use of singular value decomposition. This allows an easier analysis of structural models, for example, of a prototype vehicle, by determining the loads that the prototype is subjected to in testing. These loads can then be used on other models of that vehicle during the design process to decrease the time spent in testing while increasing design's reliability by reducing uncertainties in the applied load.**

## Introduction

CURRENTLY, differences between the physical test of a structure and finite element analysis (FEA) of its model are generally reduced by using some form of system identification, where the load is known and the model is altered after the unknowns of the system are determined. We seek to reduce those differences efficiently by modifying not the mathematical model (finite element model, generally) of a system, but instead the loading to which the structure is subjected to such that the experimental and the finite element response match in the least-squares sense.

Prototype vehicles, for instance, are subjected to particularly unknown loadings. Such vehicles generally have experimental data and are also analyzed through various theoretical and computational means. We seek to present the basis for a method to reduce the differences between the observed experimental data and the analytical results. Generally, such methods use the concept of system identification, that of treating the system as an inverse problem where the results are known and the original system is to be determined accurately. Once a system is determined, or a representative one is created, the finite element model of that system is adjusted in model updating to reduce the differences between the analytical and experimental results. We believe this can be done in a more efficient manner using the approach presented here.

System identification is a numerically intensive process. In this method the entire mathematical model of a structure is often manipulated to extract the system parameters of the mathematical model and the conditions (boundary, load, etc.) that caused the given set of experimental results. This determined system model or set of conditions is then employed or compared to another similar model. In model updating, the finite element model of a system or structure is modified to minimize differences between the behavior observed during testing and that determined by an analytical method such as the FEA. Although effective, this method can sometimes alter a model to be valid for a specific load case. This requires the model updating method to be used again for another set of loads or conditions.[1]

Load updating is more than a variation on model updating; it allows a great deal of flexibility in an analysis while preserving the mathematical model of the given system. It provides for an alteration of an applied load, leaving the mathematical model of the structure unaltered. This is faster than most forms of system identification and permits the differences between testing data and analytically determined data to be reduced in a more efficient manner. Instead of altering the model itself, as would be done in a model updating routine, one adjusts the load applied to the model to compensate for differences in how the load was actually applied. This can also be done to compensate for any limitations the model might have, for example, the use of shell vs solid elements in modeling the structure. It redresses the differences between the analysis and experimental results by determining which loads best simulate, in the context of the given analytical model, the loads that are actually applied to the structure. This results in reduced differences between the analytically determined and experimental values. Thus, it is a flexible method of minimizing differences between experimental data and FEA or determining loads for future analysis. That is, load updating can extract the equivalent loads that have been applied to a model, such that testing data can be compared with a structural model, and the appropriate computer model loading that created the results is obtained. This approach can be simpler and faster and has the added advantage of not requiring the analytical model itself to be altered, thereby avoiding the problem of tailoring a complex finite element model to a set of circumstances or conditions. The load-updating method is ideal for designs where the loads are fixed, but the structure is continuously modified.

Loads determined by load updating can be used with higher confidence on similar models. For example, if the loads for a dynamic event are determined, for example, for a vehicle test track for a vehicle suspension, the determined load could be used for similar vehicle suspension models with higher confidence than an assumed approximation of that loading.

System identification is an important step in the derivation of load updating, and an examination of the literature aids us in determining the most appropriate methodology for load updating. Johnson[2] used system informatics to derive necessary and sufficient conditions to ensure that the loads determined converge. In another paper Johnson[3] noted the difficulties in identifying a system near the applied boundary conditions and presented a method for inverse-problem solutions. In both of these papers, no examples were given.

Because difficulties in previous approaches might not be immediately evident, we examine the works of others, especially those

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works in the time domain. Through the clarification of prior methods and each method's practical use, a more appropriate and robust methodology can be determined. On the basis that a system is a superposition of signals, Johnson[4] developed a force and moment identification method using basis functions chosen to represent desired characteristics of time variations of loadings. These functions were then used in deconvolutions to determine general continuous- and discrete-time measurements. von Hoff, et al.[5] introduced the use of the transpose of the estimating function for creating stability in nonlinear algorithms as part of multichannel deconvolution. Lee and Nandi[6] further examined the use of blind-signal deconvolution to extract multiple signals. In a normal deconvolution the output signal is known. For blind-signal deconvolution the system and the system input are not known, but the system input is desired. In our case one signal (the model) is known, and another signal (the loading) is unknown; however, depending on testing, an output signal might not be fully known (i.e., we might not know the entire response of a structure). Thus, there are possibilities of using this method for load identification. Lopez and Gonzales[7] present a Levinson's algorithm for reducing the computational cost of a typical two-channel deconvolution by approximately 75%. However, Röbel[8] argued that such blind-signal deconvolutions are far more sensitive to mismatch. Röbel argued that more than a rough approximation of signal density is required for blind-signal deconvolution.

As inverse problems are ill-conditioned, Tenorio[9] made use of the idea of system regularization to converge to a likely system. In his work an assumed system was used to help regularize a set of possible solutions such that a solution with a higher confidence level was achieved. Thus, a complex system of numerous solutions can be narrowed to solutions that are more physically meaningful. Instead of regularization, Law and Fang[10] used a dynamic programming technique dependent on recursion to place bounds on ill-conditioned forces to increase accuracy of identified moving forces in the time domain. Revisiting the concept of regularization, Kirkeby et al.[11] presented the case for frequency-dependent regularization to attenuate peaks in frequency response of deconvolution problems. Similarly, Neelamani et al.[12] proposed a wavelet-based regularized deconvolution for such problems.

For determination of parameters necessary in a time-dependent system, Ackleh and Fitzpatrick[13] developed a convergence theory for determining time-dependent parameters in a parabolic system. Ackleh and Fitzpatrick also presents a stability theory for approximate methods for such systems as flexible structures. Ackleh and Fitzpatrick made use of a cubic spline to approximate the solution to the Euler–Bernoulli beam equation and examined the extraction of parameters in a noisy system.

Demonstrating the uses of load identification, Eksteen and Raath[14] reconstructed the time history of an aeroelastic load for fatigue testing by approximating the loads as quasi-static loads to create a service load simulation for laboratory test rigs. High-frequency actuators were used to simulate other dynamic effects, and these two types of loads were superposed to create a testing equivalent service load.

This paper presents load identification for the static case by means of the generally used least-squares basis. Rattray et al.[15] made use of the least-squares approach for deconvolution based on a least-squares determined impulse response function. Avitabile[1] also made use of least squares. This classic approach to system identification treated the problem as an unconstrained optimization technique that necessarily made use of the sensitivities of desired parameters with respect to changes in the system. Alternatively, Kapania and Park[16] employed the time finite element method, which discretizes time into finite elements, each with a separate time history, to determine quickly the sensitivities of these parameters. They used an iterative method to identify parameters.

There have been recent advances in the use of optics for the measurement of the displacement field of a structure during structural testing. Fu and Moosa[17] proposed in their paper to use charge-coupled device (CCD) cameras with optical data processing to determine the complete displacement field. They made use of subpixel edge detection and have proposed using CCD measurement systems for the use of health monitoring of structures. They found in laboratory environment experiments with steel I-beams a good correlation (within 5%) to measurements with dial calipers. Similarly, Liu et al.[18] have preliminary results on a proposal to use optical measurement techniques in a wind tunnel to determine the aerodynamic loads on a model. Previously using strain gauge balances, they replaced their measurements with those data-reduction methods in their work. Using two CCD cameras, they found optical measurements were less accurate than conventional strain gauge-sting balance systems, although this is preliminarily thought to be caused by the methods by which both the strain gauge balances and the optical measurements obtain their uncertainty values. This uncertainty increases with the distance between the test article and the CCD cameras. Liu et al. also note that the strain gauge balance can decouple interactions among measuring components, something which the CCDs cannot do at this time, but they note that decreasing the uncertainty measurements should be ongoing work to improve the technique's accuracy.

Here, we present a method in which traditional system identification and model updating are not performed. Instead the loads on the finite element model that would create equivalent displacements or stresses are identified by various means, thus saving a large amount of computational effort. Using techniques based on classical least-squares methods, we present the basis for finite element load updating and its sample application through the use of computer simulation.

This allows an easier analysis of structural models, such as a prototype vehicle, by determining the loads that prototype is subjected to in testing. These loads can then be used on similar models during the design process to decrease the time spent in testing without decreasing the design suitability.

## Mathematical Formulation

Although the load on the structure is not fully known, we assume that we are not working in an informational vacuum as far as the load distribution is concerned. We use this knowledge as a basis for creating an initial function that most likely describes the load. This representative function can be expressed as a function of the spatial position on the structure. For a beam this means the load is a function of the location on the beam. Thus, this enables us to assume a function of the position on the object (in this case a beam) to approximate the mean load distribution. As this is an approximation based on assumptions about the load, it is inappropriate to assume this load is an exact fit, and we allow for fluctuations about this mean function so as to allow it to model the loading conditions accurately. When we change the mathematical model into a set of finite elements, we would ideally assume that each of the fluctuations has a mean value of zero. This enables us to model accurately the load acting over an element in the finite element model by using a simple superposition of the assumed average load and polynomials representing the fluctuations. This is seen in example with a beam in Fig. 1.

We begin with the assumption of an average function value of $f_0(x)$ over the simply supported beam. This value is an approximation of the beam loading in magnitude and form. There are also

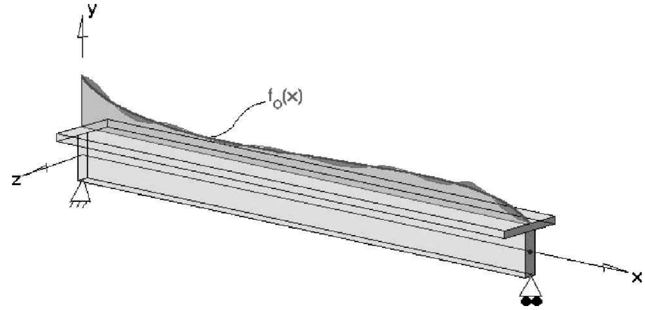


**Fig. 1 Example simply supported T-stiffener beam subjected to an unknown load, approximated as $f_0(x)$.**

slight fluctuations about this mean just shown. These fluctuations are what allow an accurate modeling of the load for creating an accurate finite element model. Thus, we end up with a loading function over an element of

$$f(x) = f_0(x) + \sum_{i=1}^{n} c_i P_i(x) \tag{1}$$

where the $c_i$ are unknown weighting coefficients and $P_i$ are known functions with zero mean. The assumed average function $f_0(x)$ can be written as a superposition of the polynomials that covers linear fluctuations across an element. In this way the $f_0(x)$ term can be explicitly eliminated, as it is implicit in the polynomials, and so Eq. (1) becomes

$$f(x) = \sum_{i=1}^{n} c_i P_i(x) \tag{2}$$

where $c_i$ are the coefficients that now compensate for the lack of an average load function $f_0(x)$. This assumption is predicated on the selection of a set of functions that include a linear approximation. Without functions that include this approximation, the form in Eq. (1) should be used. This is because linear or discontinuous loads will not be able to be approximated correctly as a result of Gibbs phenomenon. The number $n$ of functions over an element will allow a more accurate superposition prediction of the element loading, that is, the larger the number of functions used, the finer the resolution of the load. This has its limits as a result of the accuracy of the element used; thus, the inclusion of large numbers of zero-mean polynomials for fluctuations has a diminishing return that is inversely proportional to the order of the element. In a practical application it was found that the practical limit on the number of polynomials used over an element was reached much earlier than expected.

In general, for our finite element modeling, we begin with the following relation of the stiffness $\boldsymbol{K}$ and displacement vector $\boldsymbol{u}$ to the force vector $\boldsymbol{f}$:

$$Ku = f$$

or we can explicitly solve for the displacement components $u_j$ as

$$u_j = K_{ij}^{-1} f_i$$

To simplify the computations, we will use only the degrees of freedom for which $u_j$ is known. Additionally, we propose the force vector can be expressed as a matrix $\boldsymbol{F}$ multiplied with a vector of coefficients. The force coefficient matrix $\boldsymbol{F}$ is such that

$$\{\boldsymbol{f}\} = [\boldsymbol{F}]\{\boldsymbol{c}\} \tag{3}$$

$$u = [\boldsymbol{K}]^{-1}[\boldsymbol{F}]c\colon N \times M$$

where $N$ is the number of coefficients and $M$ is the number of degrees of freedom.

We use least-squares approximations to minimize the error between the measured values of the displacements obtained in an experiment and those found by finite element analysis. This is done by changing the values of $c_i$, which would minimize these errors. For the displacements we minimize the square of the difference between the exact displacement vector $U$ (given or from data) and the displacement vector $V$ calculated by the FEA:

$$\min \|(\boldsymbol{U} - \boldsymbol{V})\|_2^2$$

To minimize this, we will need the gradient of the function, which is

$$\nabla\Big[\|(\boldsymbol{U} - \boldsymbol{V})\|_2^2\Big]_i^e = 2(U_i - V_i)\cdot(\partial_h V_i)^T$$

The gradient is derived at an elemental level, but is assembled to represent a global gradient. This gradient is related to the sensitivities of the displacements (or strains) with respect to each of the coefficients $c_i$. This allows the optimal coefficients to be determined iteratively by changing the value of each of the coefficients in a stepwise manner that will reduce the difference. The gradient is used to ensure that the steps taken are in a direction which minimizes the differences between the measured and calculated values. As the coefficients are changed, the differences between the calculated results and experimental results are changed as well until these differences are at a local minimum point. This is determined by the magnitude of the vector of those differences. Thus the gradient can be used in the steepest descent method, or similar methods, to continue altering the coefficients.

Taking the load distribution over a given element, say the $e$th, the elemental load $p^e(\bar{x})$ is represented as

$$p^e(\bar{x}) = \sum_{i=1}^{i=m} c_i^e P_i(\bar{x}) \tag{4}$$

where $P_i$ are the basis functions, taken to be the integrated Legendre polynomials; $m$ is the number of basis functions considered for each element, taken to be 4 in this derivation; $\bar{x}$ is the local coordinate system for a given beam element; and $c_i^e$ are the $m$ unknown coefficients for each element. Thus, the total number of unknowns in the system that has not been reduced for known boundary conditions will then be $m * N_e$, where $N_e$ is the number of elements.

The consistent load vector for the $e$th element $q^e$ is given as

$$\boldsymbol{q}^e = \boldsymbol{F}^e \boldsymbol{c}^e \tag{5}$$

Here the matrix $\boldsymbol{F}^e$ is defined by

$$F_{i,j}^e = \int_0^{l_e} N_i(\bar{x}) P_j(\bar{x})\, d\bar{x}, \qquad i = 1, 4, \qquad j = 1, m \tag{6}$$

where $l_e$ is the length of the $e$th element and $N_i(\bar{x})$ is the $i$th shape function for the beam element.

The global load vector $\boldsymbol{f}$ is obtained by assembling the load vector for individual elements and was shown in Eq. (3). For the unrestrained system $\boldsymbol{F}$ is a matrix of $4N_e$ rows and $mN_e$ columns for a beam element, where $mN_e$ is the same as the number of total unknowns in the vector of unknown coefficients $\boldsymbol{c}$. The number of rows in matrix $\boldsymbol{F}$ will reduce by the number of restraining boundary conditions. The $\boldsymbol{F}$, for three elements of four coefficients per element, is obtained from the individual element matrices $\boldsymbol{F}^e$ as seen in Eq. (7):

$$\boldsymbol{F} = \begin{bmatrix} F_{11}^1 & F_{12}^1 & F_{13}^1 & F_{14}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F_{21}^1 & F_{22}^1 & F_{23}^1 & F_{24}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F_{31}^1 & F_{32}^1 & F_{33}^1 & F_{34}^1 & F_{11}^2 & F_{12}^2 & F_{13}^2 & F_{14}^2 & 0 & 0 & 0 & 0 \\ F_{41}^1 & F_{42}^1 & F_{43}^1 & F_{44}^1 & F_{21}^2 & F_{22}^2 & F_{23}^2 & F_{24}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_{31}^2 & F_{32}^2 & F_{33}^2 & F_{34}^2 & F_{11}^3 & F_{12}^3 & F_{13}^3 & F_{14}^3 \\ 0 & 0 & 0 & 0 & F_{41}^2 & F_{42}^2 & F_{43}^2 & F_{44}^2 & F_{21}^3 & F_{22}^3 & F_{23}^3 & F_{24}^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{31}^3 & F_{32}^3 & F_{33}^3 & F_{34}^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{41}^3 & F_{42}^3 & F_{43}^3 & F_{44}^3 \end{bmatrix} \tag{7}$$

The vector of unknown coefficients $\boldsymbol{c}$ will be obtained using the method of least squares. We will minimize the square of the error between the measured response and the response calculated using the finite element method.

Let $\boldsymbol{u}_m$ represent the measured response and $\boldsymbol{u}$ represent the calculated response. The square of the error $E$ is given as

$$E = \tfrac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}_m\|_2 = \tfrac{1}{2}\boldsymbol{e}^T\boldsymbol{e} \tag{8}$$

Here the vector $\boldsymbol{e}$ is defined as $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_m$

Assume that the finite element model for the system is given as

$$\boldsymbol{Ku} = \boldsymbol{f} = \boldsymbol{Fc} \tag{9}$$

Here $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{u}$ is the vector of finite element degrees of freedom, $\boldsymbol{f}$ is the load vector, $\boldsymbol{F}$ is the force matrix just derived from the polynomials, and $\boldsymbol{c}$ is the vector of unknown weighting coefficients. Because the response can be measured only at a limited number of points, the order of vectors $\boldsymbol{u}$ and $\boldsymbol{u}_m$ will be different. If we consider $\boldsymbol{u}_m$ to be fully populated, then $\boldsymbol{u}$ contains more information than can be used and must be reduced to the same dimension. Thus, the two vectors are related to each other by a matrix $\boldsymbol{T}$ as

$$\begin{aligned} \boldsymbol{u}_m &= \boldsymbol{T}\boldsymbol{u} \\ &= \underbrace{\boldsymbol{T}\boldsymbol{K}^{-1}\boldsymbol{F}}_{G}\boldsymbol{c} \\ &= \boldsymbol{G}\boldsymbol{c} \end{aligned} \tag{10}$$

The partial derivative of the error vector $\boldsymbol{e}$ with respect to $c_i$, the $i$th element of unknown coefficient vector $\boldsymbol{c}$, can be written as

$$\frac{\partial \boldsymbol{e}}{\partial c_i} = \frac{\partial \boldsymbol{u}}{\partial c_i} = \boldsymbol{G}_i \tag{11}$$

Here $\boldsymbol{G}_i$ is the $i$th column of the matrix $\boldsymbol{G}$.

The magnitude of the error $\boldsymbol{e}$ can be minimized either using any of the standard minimizing schemes such as the method of steepest descent or the conjugate gradient method. Alternatively, the error $E$ can be minimized using the condition

$$\frac{\partial E}{\partial c_i} = 0 \tag{12}$$

The preceding equation is a set of $mN_e$ equations. The partial derivative of the error with respect to $c_i$ can be written as

$$\begin{aligned} \frac{\partial E}{\partial c_i} &= \frac{1}{2}\left[\boldsymbol{e}^T \frac{\partial \boldsymbol{e}}{\partial c_i} + \left(\frac{\partial \boldsymbol{e}}{\partial c_i}\right)^T \boldsymbol{e}\right] \\ &= \boldsymbol{G}_i^T \boldsymbol{e} \end{aligned} \tag{13}$$

Substituting this relation in the minimization condition leads to a set of $mN_e$ equations, given as

$$\boldsymbol{G}^T \boldsymbol{G} c = \boldsymbol{G}^T \boldsymbol{u}_m \tag{14}$$

This creates a symmetric square matrix system that is solvable through an iterative method such as the conjugate gradient method, as well as lending itself to possible "direct solution" of the system through any matrix solver routine.

An analysis to obtain the condition number of the matrix $\boldsymbol{G}^T\boldsymbol{G}$ in Eq. (14) was performed. This number is the equivalent to the square of the conditioning number of $\boldsymbol{G}$. This analysis was conducted for $\boldsymbol{G}^T\boldsymbol{G}$ of the one-, three-, and eight-element cases. The progression of the condition number is shown in Table 1.

The conditioning number of the system, shown in Eq. (14), is such that it is better to solve Eq. (14) using a biconjugate gradient system or by using the so-called singular value decomposition (SVD) approach.

**Table 1 Comparison of condition number vs number of elements where "high order" or "low order" denotes four or two polynomials per element, respectively**

| Number of elements | Condition number |
|---|---|
| 1 (low order) | 1.67e4 |
| 3 (low order) | 1.24e7 |
| 3 (high order) | 6.49e19 |
| 8 (low order) | 1.76e18 |

## Examples

In these examples the beam used was a T-section, the generic representative dimensions determined from examination of a sample vehicle structure. The beam was given specific boundary conditions; in the initial case it was simply supported with a varying, distributed load.

The set of integrated Legendre polynomials was selected for the perturbation equations as they form a complete set of mutually orthogonal functions, although they are not, as ideally, of zero mean. As just mentioned, the first two integrated Legendre polynomials allow us to eliminate the average function in the assumption of the load as those polynomials are linear and thus will compensate for the average function. Initially, each element had four weighting coefficients for four integrated Legendre polynomials per element. The first several of these polynomials are shown in Fig. 2.

For our proof of method, we required an input of a known load. These were created for our load updating examples from a given FEA case, where in practice these would instead come from testing data for the given FEA model.

To perform a comparison between the given and extracted load case, the given displacement (i.e., the measured values from our known hypothetical experiment) can be input into a simple FEA code. In this way the hypothetical structure's stiffness matrix and consistent load vector are input for eventual comparison. This built-in example simulates an ideal experimental system that we wish to determine the loading on. This input of the known loads is only for comparison purposes and is not used in the load extraction routines as these routines are used to determine the applied load. As we are using a given FEA example with a simple built-in FEA code, we made use of a given sample loading (e.g., a uniform load, sine load, etc.) to create the input of the consistent load vector and thus create displacements. These FEA-determined displacements are used in the load extraction of this load updating example. Initial examples used unit elemental lengths, with nondimensional unit-distributed loads. This FEA example is only for simulation of an ideal experimental set and not used by the load updating routines, only selected analytical displacements are used to emulate a set of testing data.

The beam was then converted to nonunit element lengths, with a T-stiffener cross section, and both uniform and sine wave loadings were applied. The final beam length in all examples was 1 m. The cross section had a moment of inertia $I_{zz}$ of 2.902e-7 m$^4$. The Young's modulus $E$ for the material of the beam was for aluminum 7.102e7 MPa.

Initially, it was found that with four polynomials per element, although an exact solution could be found to represent the force by coefficients, the steepest descent did not converge to that solution. Thus, there was with an ill-conditioned problem caused by the increase in polynomials; the method was giving spurious results because of increased numbers of local minima to which the least-squares routine could converge that would satisfy the convergence criteria. This was found in both the one- and three-element cases and can be seen in Fig. 3, where the representative determined load is far from what was applied. The deflection curve for the actual applied load and the incorrect determined load are nearly identical because of minimization with respect to differences between

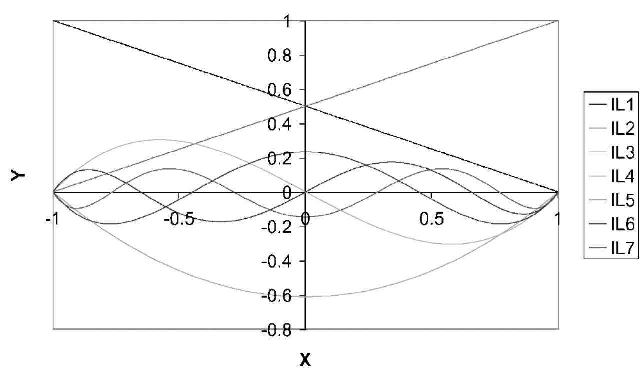


**Fig. 2 First several integrated Legendre polynomials.**

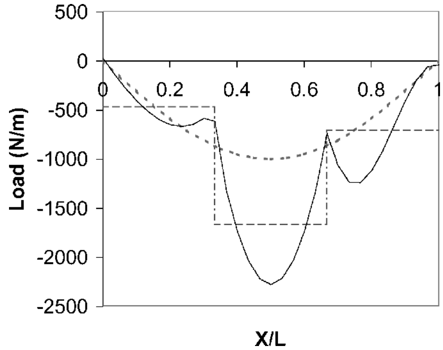


**Fig. 3 Applied and extracted loads case for the higher-order three-element case of a T cross-section cantilever beam subjected to an applied sine load: - - - -, applied force; ——, extracted; and – - - -, element average.**

"experimental" and "analytical" deflections. However, although the deflection curves are similar, the load vectors are not. In this example the known load vector can be compared to the equivalent load vector. However this is not possible from a practical standpoint as the applied load vector is unknown.

It was hypothesized that the higher order of the increased number of polynomials representing the load can represent mode shapes to which the element cannot respond. That is, the beam element used could not respond properly to the additional modes provided by the higher-order polynomials; thus, the order of polynomials was decreased. This gave only a linearly varying load approximation over an element, yet the accuracy of the approximation of the determined loads increased.

Damped least-squares regularization was applied to the system in Eq. (14), but it was found that slight changes in the regularization parameter were found to result in widely varying extracted loads, none of which were correct. Using the pesudoinverse from SVD has temporarily mitigated the immediate need for regularization, and it was no longer implemented in the current program revision.

Also affecting the load extraction is the simulation of noise in the system. Noise was added to the system response through the use of a Gaussian deviant system with a mean of approximately zero and a standard deviation of approximately one. This was then scaled to the order of magnitude of the measurements and applied as a percentage of this magnitude to the given measurements. Thus, 10% noise would be a random number times 10% times the average magnitude of the vector, added to the original element number.

Given this introduction of noise, the lack of regularization, and the conditioning number of the base system that renders methods such as conjugate gradient ineffective, we instead introduced another method for solution, in this case that of SVD. This allowed a method of determining the spectral values of the system, a method for determining the conditioning number of the system examined, and a method for a pseudoinverse.

In using SVD to solve the initial problem via pseudoinverse, it is known that if one alters the eigenvalues given by singular value decomposition it can have the effect of rendering SVD into a least squares solving routine.[19] Thus, if those altered eigenvalues are set to zero this effectively damps high frequencies, operationally attenuating the system's response to those values. The error between the determined force vector and the applied force vector can be found by

$$\%\text{Error} = \frac{\| F_{\text{applied}} - F(c)_{\text{calculated}} \|_2}{\| F_{\text{applied}} \|_2} \times 100\% \qquad (15)$$

if one knows the applied load to the system ($F_{\text{applied}}$). If one solves the system with no noise applied to the input system response and without reducing any eigenvalues, the error is quite low at 0.01%. Using the same system, if one reduces determined eigenvalues below 1E-10 to zero the error goes to 0.8%. The error similarly changes such that more noise in the applied load requires more attenuation of rapid spacial variations in the load. These variations represent the rapid shifting of the applied load caused by the superposition of randomness to the overall load vector. This attenuation is done by zeroing more eigenvalues determined in the SVD of the system. There is a diminishing return though. If one zeros all but one of the eigenvalues, the full shape of the applied load cannot be captured. In this case, with a simple sine-wave shape, the load profile is captured by zeroing all but two of the lowest value eigenvalues for all levels of noise. At low levels of noise (noise was 0–20% of the magnitude of the load vector), this results in about 20% error levels (depending on the seed value of the random noise generator). However, if the noise is increased to as much as 90% of the average magnitude of the load vector the error is only 30–33% (again depending on the seed). If one does not attenuate the higher frequencies, the effect on the solution found by SVD-determined pseudoinverse results in very high error (3300% or more). Thus, SVD cannot be the primary method of solution.

In general, the uniform or discontinuous load distribution is not easy to be identified as such distributions need a large number of superpositioned polynomials to simulate the load, a phenomenon known as the Gibbs phenomenon. Because of the selection of the integrated Legendre polynomials in this case, a uniform load could be accuratly simulated by a superposition of polynomials.

Figures 3 and 4 show the difference in the extracted loads using lower-order polynomials and the reduced higher-order polynomials, respectively. Thus, an eight-element beam system was created so that the linear approximation would be more accurate, much in the way the accuracy increases in an h-version finite element approach. The extracted load and the applied sine load are shown in Fig. 5 for this eight-element beam with a lower number (lower-order) of polynomials per element. Showing excellent results, a similar trend was exhibited in the solution for a uniform load. The deflection curves for the calculated and actual applied loads are shown in Fig. 6.

The difficulty in many of these convergence studies is that the loads on the elements which are on the boundary seem to converge at

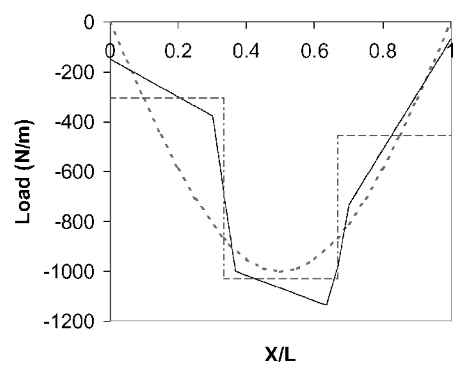


**Fig. 4 Applied and extracted loads for the lower-order three-element case of a T cross-section cantilever beam subject to an applied sine load over the nondimensionalized length. The average load across the element of the extracted load is given for illustrative purposes: - - - -, applied force; ——, extracted; and – - - -, element average.**

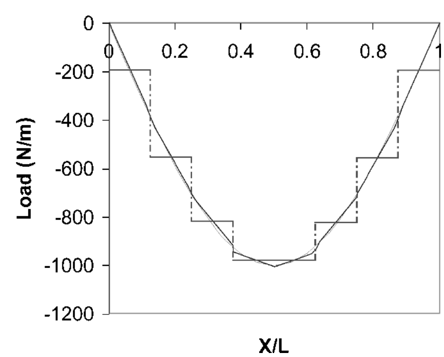


**Fig. 5 Applied and extracted loads for the lower-order eight-element case of a T cross-section cantilever beam subjected to an applied sine load. The element average is a piecewise representation of the average extracted load of each element: ......, applied force; ——, extracted; and – - –, element average.**

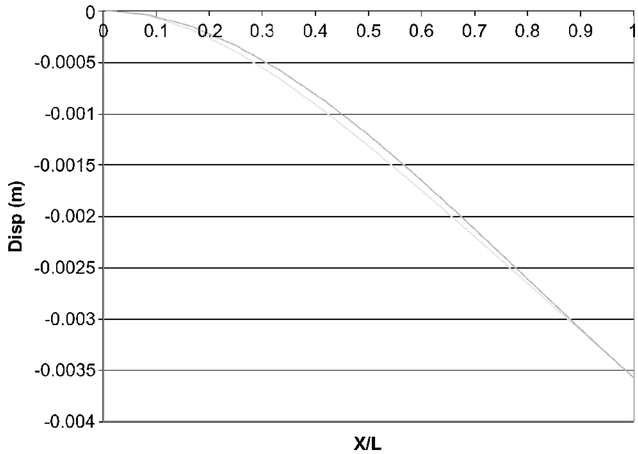


**Fig. 6 Deflection curve of the cantilever beam for one- and eight-element load cases subject to a sine load for the given and extracted loads from a uniform load applied to the cantilevered inverse T-stiffener beam. The eight-element Hermite displacement and calculated curves overlap as does the single-element Hermite displacement and calculated curves: ——, eight-element FEA displacement; ——, eight-element calculation; ——, one-element FEA; ……, one-element low order.**

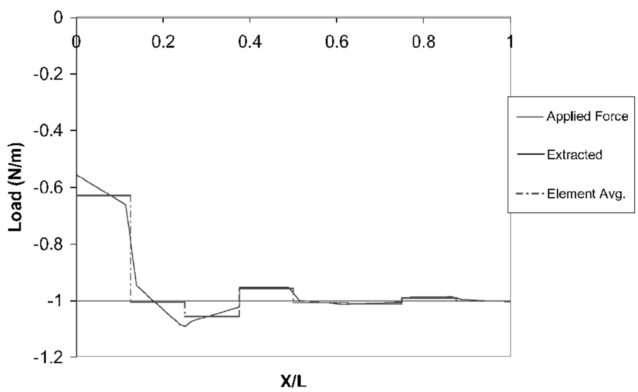


**Fig. 7 Example from the eight-element uniform load example showing the difference in convergence between the elements close to the applied boundary condition ($x/L = 0$) and the remaining elements after 3000 iterations.**

a much slower rate than the rest of the structure. Possible solutions to avoid this include making smaller elements near the boundaries and increasing the number of elements so that this effect is reduced, as mentioned in the h-version option for increasing accuracy. Changing the model such that there are smaller elements at the boundaries is not unlike model updating in that the computer model is now being adjusted for the correct simulation of the actual boundary conditions. This effect is also noted in some of the literature.[3] It can be seen in Fig. 7 where a uniform load has been applied to the cantilever beam. In this instance, although a couple of thousands of iterations have been completed and though many of the coefficients are near their final values, the values in the two elements nearest to the boundary condition (the left two elements in Fig. 7) are still relatively far from convergence. Given the rate of convergence in this model, several thousand more iterations would most likely be needed for convergence to the applied load.

Use of singular value decomposition as a means to obtain an initial guess to the load appears to be more useful. Once a solution is found using the pseudoinverse, even with noise, but with appropriate damping used, the steepest descent solver routines seem to converge to an answer that has a percent error on a similar order to the noise level, as seen in Table 2.

The core of the needed numerical routines were taken from the LAPACK library and used in the matrix manipulations needed in solving the inverse problem. The option of solving the inverse problem by conjugate gradient method was initially included, thus making the number of solution methods to three; however, after the conditioning analysis conjugate gradient routines were replaced with routines for SVD. The present approach can be extended to solve problems in two dimensions, and eventually problems in which the load is a function of time also.

**Table 2 Comparison of noise level vs error after use of SVD with steepest descent**

| Noise level, % | Initial error results, % |
|---|---|
| 5 | 6 |
| 10 | 14 |
| 15 | 23 |

## Conclusions

A method for reducing the differences between a finite element model and experimental testing has been presented that shows that extensive computation to reduce theses differences by adjusting the system or the original finite element model might not be necessary. The determination of loads, and then adjustment of those determined loads, might instead reduce those differences without the altering of the original finite element model. This prevents the need to determine numerous coefficients while leaving a given FEA system undisturbed. The introduction of SVD for the computation of a pseudoinverse and the conditioning number of the system were found to be a useful in the computation of an initial guess for use in the steepest descent method. The method's capabilities in the presence of noise applied to the given system displacement measurements was also examined and quantified with respect to changes in the eigenvalues found by SVD and used in the pseudoinverse. Although results found that reducing the class of polynomial used in simulating the load over an element increased linear accuracy, it remains to be seen if regularization would instead allow a higher order of polynomial to be used, negating the use of increased numbers of elements each using with a lower order of polynomials to represent the load. Comparison of determined loads with known values was found to be favorable in examined problems.

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